A Logical Approach to Discrete Math

Propositional Calculus

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Propositional expressions are type boolean: $(p \wedge q \Rightarrow r) : \mathbb{B}$

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Example: equals(n, m) is a predicate. equals($7, 11$) returns false.

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Example: equals(n, m) is a predicate. equals$(7, 11)$ returns false.
The type of function equals is $\mathbb{Z} \times \mathbb{Z} \to \mathbb{B}$

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Example: odd.n is a predicate. odd.5 returns true. odd.6 returns false. The type of function odd is $\mathbb{Z} \to \mathbb{B}$

Example: equals(n, m) is a predicate. equals$(7, 11)$ returns false. The type of function equals is $\mathbb{Z} \times \mathbb{Z} \to \mathbb{B}$

Example: sum(n, m) is not a predicate. sum$(7, 11)$ returns 18.

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Example: equals(n, m) is a predicate. equals$(7, 11)$ returns false.
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The type of function sum is $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

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Universal quantification with $\forall$ and existential quantification with $\exists$ are predicates.

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Example: $(\Sigma i \mid 0 \leq i < n : b[i])$ is <u>not</u> a predicate.
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(9.2) Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
b	23	14	-6	5	-7	-13	-23	-4	19	-2

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$$(\forall i \mid 4 \leq i < 8 : b[i] < 0)$$

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$$
\begin{aligned}
& (\forall i \mid 4 \le i < 8 : b[i] < 0) \\
= \quad & \langle (9.2) \rangle \\
& (\forall i \mid : 4 \le i < 8 \Rightarrow b[i] < 0)
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This range is over all 10 values.

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Notation: $(\star x \mid : P)$ means $(\star x \mid true : P)$.

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

Universal quantification trades with implication.

Theorems of the Predicate Calculus

Universal quantification.

Notation: $(\star x \mid : P)$ means $(\star x \mid \mathit{true} : P)$.

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.3) **Trading:**

(a) $(\forall x \mid R : P) \equiv (\forall x \mid : \neg R \vee P)$

(b) $(\forall x \mid R : P) \equiv (\forall x \mid : R \wedge P \equiv R)$

(c) $(\forall x \mid R : P) \equiv (\forall x \mid : R \vee P \equiv P)$

(9.4) **Trading:**

(a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$

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(d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$

(9.4.1) **Universal double trading:** $(\forall x \mid R : P) \equiv (\forall x \mid \neg P : \neg R)$

Prove (9.4a) Trading: $(\forall x \mid Q \land R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$

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$$(\forall x \mid Q \wedge R : P)$$

$=$ $\langle$(9.2) Trading$\rangle$

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Proof

$$
\begin{array}{ll}
 & (\forall x \mid Q \wedge R : P) \\
= & \langle \text{(9.2) Trading} \rangle \\
 & (\forall x \mid : Q \wedge R \Rightarrow P)
\end{array}
$$

Prove (9.4a) Trading: $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$

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$$
\begin{array}{ll}
& (\forall x \mid Q \wedge R : P) \\
= & \langle (9.2) \text{ Trading} \rangle \\
& (\forall x \mid : Q \wedge R \Rightarrow P) \\
= & \langle (3.65) \text{ Shunting} \rangle
\end{array}
$$

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$(\forall x \mid Q \wedge R : P)$

$=$ ⟨(9.2) Trading⟩

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$(\forall x \mid Q : R \Rightarrow P)$ //

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Proof

$(\forall x \mid \neg P : \neg R)$

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$(\forall x \mid : \neg P \Rightarrow \neg R)$

Prove (9.4.1) Universal double trading: $(\forall x \mid R : P) \equiv (\forall x \mid \neg P : \neg R)$

Proof

$$
\begin{aligned}
& (\forall x \mid \neg P : \neg R) \\
= \quad & \langle (9.2)\ \text{Trading} \rangle \\
& (\forall x \mid : \neg P \Rightarrow \neg R) \\
= \quad & \langle (3.61)\ \text{Contrapositive} \rangle
\end{aligned}
$$

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Proof

$$
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$(\forall x \mid R : P)$ //

(9.5) **Axiom, Distributivity of $\vee$ over $\forall$:** Provided $\neg occurs(`x', `P')$,
$P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

(9.6) Provided $\neg occurs(`x', `P')$, $(\forall x \mid R : P) \equiv P \vee (\forall x \mid : \neg R)$

(9.7) **Distributivity of $\wedge$ over $\forall$:** Provided $\neg occurs(`x', `P')$,
$\neg(\forall x \mid : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$

(9.8) $(\forall x \mid R : true) \equiv true$

(9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$

(9.10) **Range weakening/strengthening:** $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$

(9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$

(9.12) **Monotonicity of $\forall$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$

(9.13) **Instantiation:** $(\forall x \mid : P) \Rightarrow P[x := E]$

(9.16) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

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<u>Example</u>

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<u>Example</u>
$p \Rightarrow p \vee q$ is a theorem.

(9.16) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

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Example

$p \Rightarrow p \vee q$ is a theorem.

Therefore, by (9.16), $(\forall p, q \mid : p \Rightarrow p \vee q)$ is a theorem.

(9.16) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

(9.16) is called "Implicit universal quantification."

Example

$p \Rightarrow p \lor q$ is a theorem.

Therefore, by (9.16), $(\forall p, q \mid : p \Rightarrow p \lor q)$ is a theorem.

In other words, $p \Rightarrow p \lor q$ is true in all states.

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$p \vee q$ is not a theorem.

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Therefore, by (9.16), it is <u>not</u> the case that $(\forall p, q \mid : p \lor q)$ is a theorem.

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$p \Rightarrow p \vee q$ is a theorem.

Therefore, by (9.16), $(\forall p, q \mid : p \Rightarrow p \vee q)$ is a theorem.

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The theorem $p \Rightarrow p \vee q$ is "Implicitly universally quantified."

Example

$p \vee q$ is not a theorem.

Therefore, by (9.16), it is not the case that $(\forall p, q \mid : p \vee q)$ is a theorem.

In other words, $p \vee q$ is not true in all states.

Existential quantification.

(9.17) **Axiom, Generalized De Morgan:** $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.18) **Generalized De Morgan:**

(a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$

(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$

(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.20) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$

(9.20.1) **Existential double trading:** $(\exists x \mid R : P) \equiv (\exists x \mid P : R)$

(9.20.2) $(\exists x \mid : R) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\exists x \mid R : P))$

(9.21) **Distributivity of $\wedge$ over $\exists$:** Provided $\neg occurs(`x', `P')$,

$P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$

(9.22) Provided $\neg occurs(`x', `P')$, $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid : R)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

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(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.19) Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
b	19	-26	17	3	42	-19	8	35	14	-30

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.19) Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
b	19	-26	17	3	42	-19	8	35	14	-30

$$(\exists i \mid 4 \leq i < 8 : b[i] < 0)$$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.19) Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
b	19	-26	17	3	42	-19	8	35	14	-30

$$(\exists i \mid 4 \leq i < 8 : b[i] < 0)$$

$$= \quad \langle (9.19) \rangle$$

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b	19	-26	17	3	42	-19	8	35	14	-30

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\begin{aligned}
& (\exists i \mid 4 \leq i < 8 : b[i] < 0) \\
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	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
b	19	-26	17	3	42	-19	8	35	14	-30

$$(\exists i \mid 4 \leq i < 8 : b[i] < 0)$$

$$= \quad \langle (9.19) \rangle$$

$$(\exists i \mid : 4 \leq i < 8 \wedge b[i] < 0)$$

This range is over all 10 values.

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

Existential quantification trades with conjunction.

(9.23) **Distributivity of $\vee$ over $\exists$:** Provided $\neg occurs(`x\text{'}, `P\text{'})$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$

(9.24) $(\exists x \mid R : false) \equiv false$

(9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$

(9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$

(9.26.1) **Body weakening/strengthening:** $(\exists x \mid R : P \wedge Q) \Rightarrow (\exists x \mid R : P)$

(9.27) **Monotonicity of $\exists$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$

(9.28) **$\exists$-Introduction:** $P[x := E] \Rightarrow (\exists x \mid : P)$

(9.29) **Interchange of quantification:** Provided $\neg occurs(`y\text{'}, `R\text{'})$ and $\neg occurs(`x\text{'}, `Q\text{'})$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.30) Provided $\neg occurs(`\hat{x}\text{'}, `Q\text{'})$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem.

(9.29) **Interchange of quantification:** Provided $\neg occurs(`y', `R')$ and $\neg occurs(`x', `Q')$,
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Example

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Example

P : The predicate loves(x, y), person x loves person y

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R : The set of x values $\{a, b, c\}$

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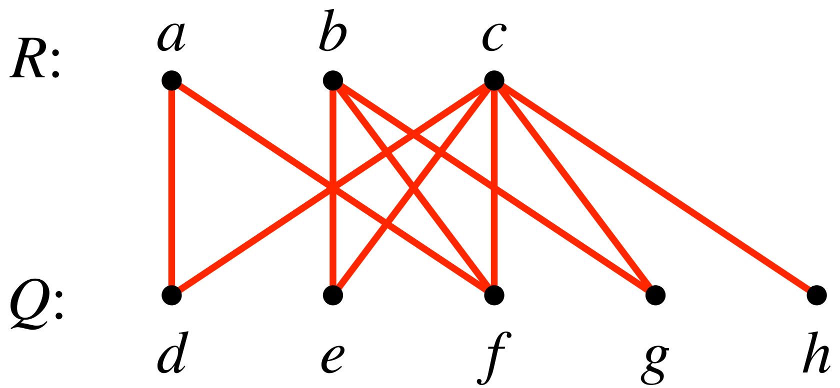

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Example

P : The predicate loves(x, y), person x loves person y

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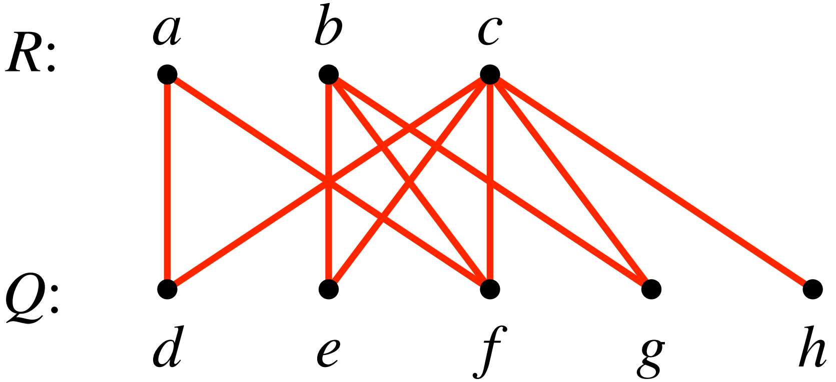
a loves d and f.



A Logical Approach to Discrete Math

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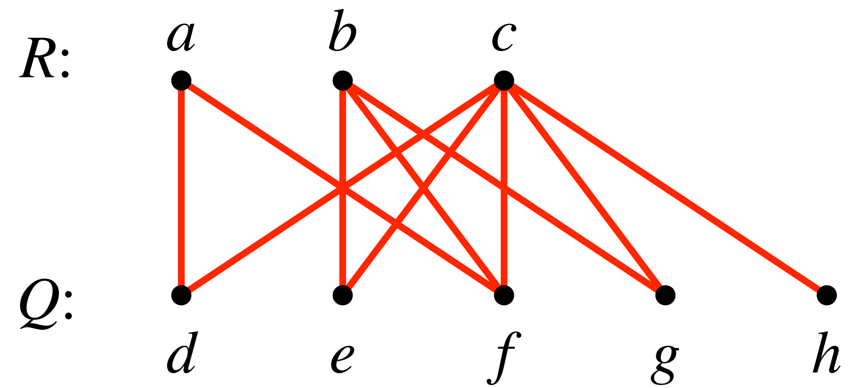
a loves d and f.

b loves e, f and g.



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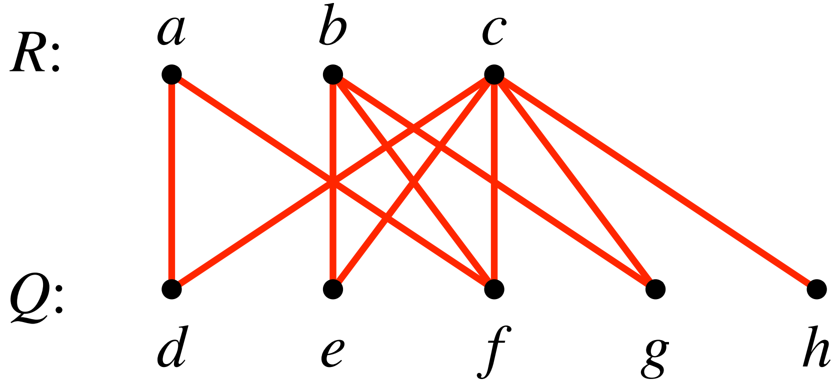
a loves d and f.
b loves e, f and g.
c loves d, e, f, g and h.



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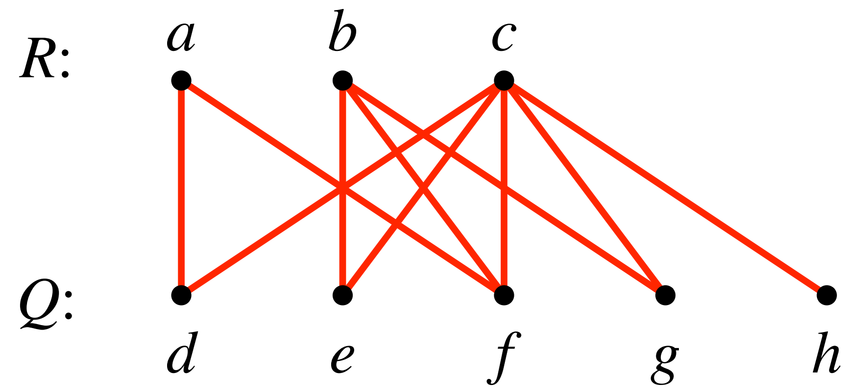
$(\exists x \mid R : (\forall y \mid Q : P))$ means



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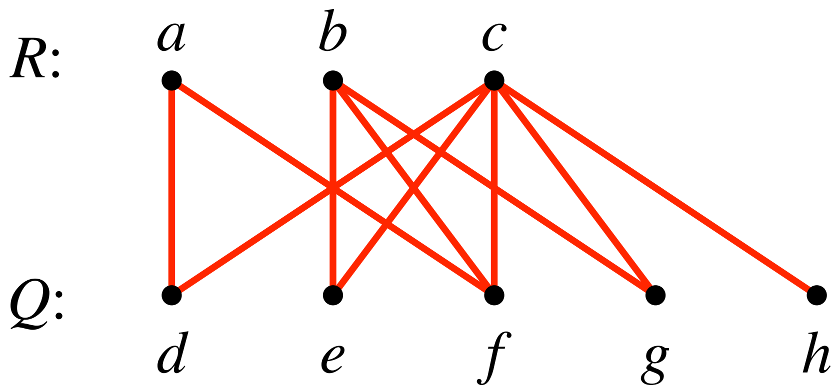

$(\exists x \mid R : (\forall y \mid Q : P))$ means "Someone in R loves everyone in Q."

A Logical Approach to Discrete Math

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<u>Example</u>

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R : The set of x values $\{a, b, c\}$

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a loves d and f.

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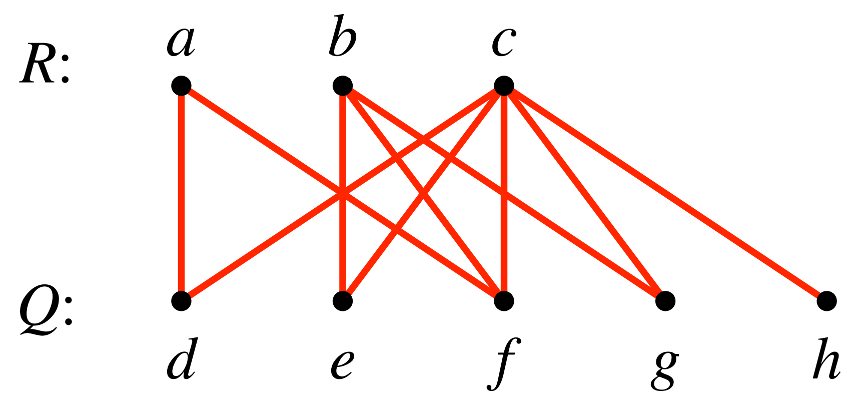

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a loves d and f.
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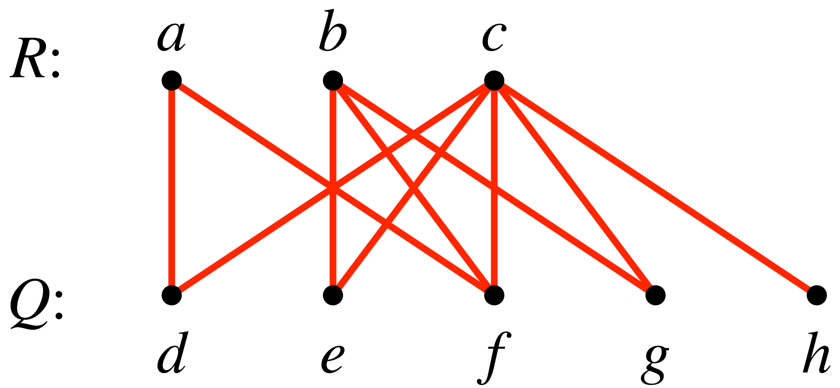

$(\exists x \mid R : (\forall y \mid Q : P))$ means "Someone in R loves everyone in Q."

$(\forall y \mid Q : (\exists x \mid R : P))$ means "Everyone in Q is loved by someone in R."

(9.29) **Interchange of quantification:** Provided $\neg occurs(`y', `R')$ and $\neg occurs(`x', `Q')$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

Counterexample for

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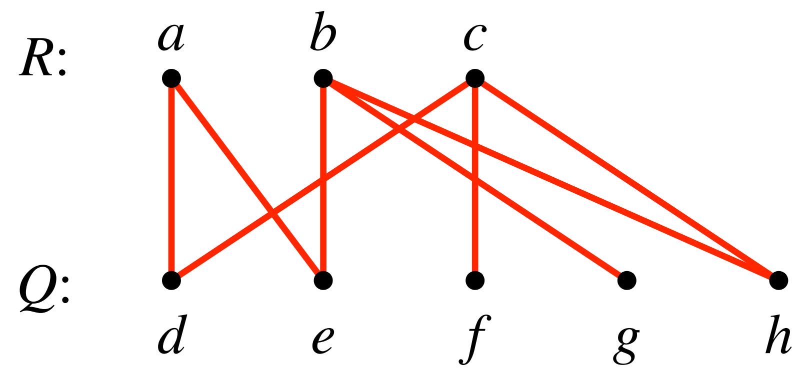

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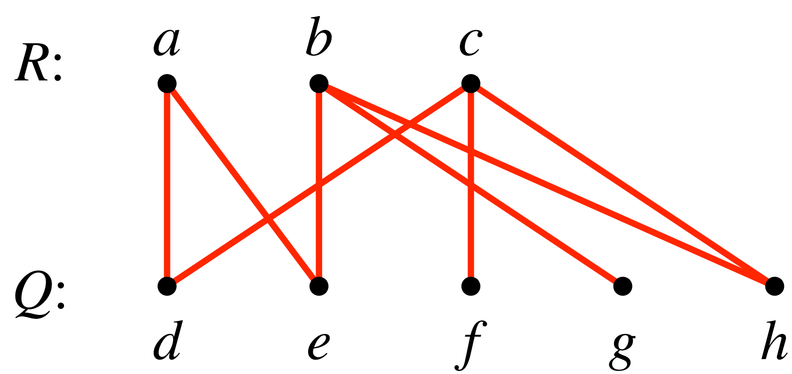
a loves d and e.



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Counterexample for

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a loves d and e.

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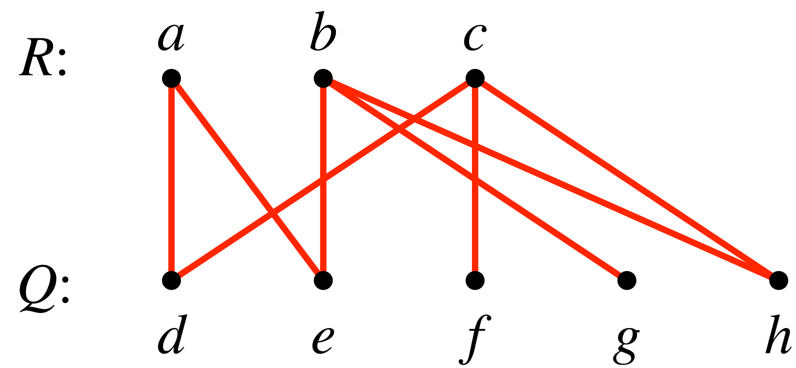

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Counterexample for

$$(\forall y \mid Q : (\exists x \mid R : P)) \Rightarrow (\exists x \mid R : (\forall y \mid Q : P))$$

a loves d and e.

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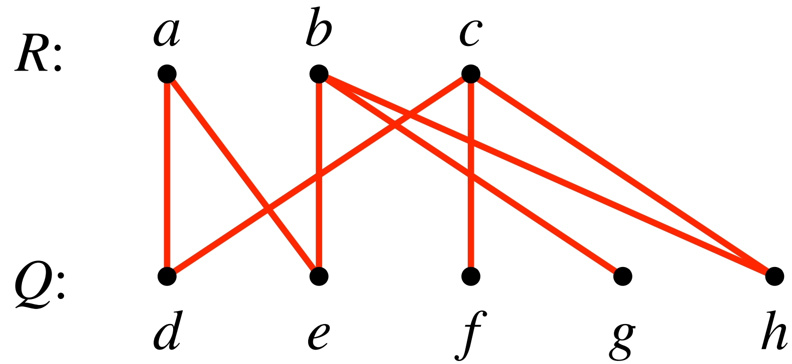

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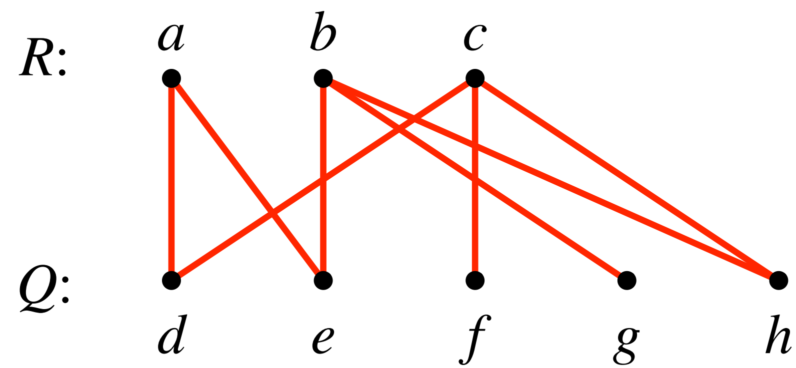

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$(\forall y \mid Q : (\exists x \mid R : P))$ is true. Everyone in Q is loved by someone in R.

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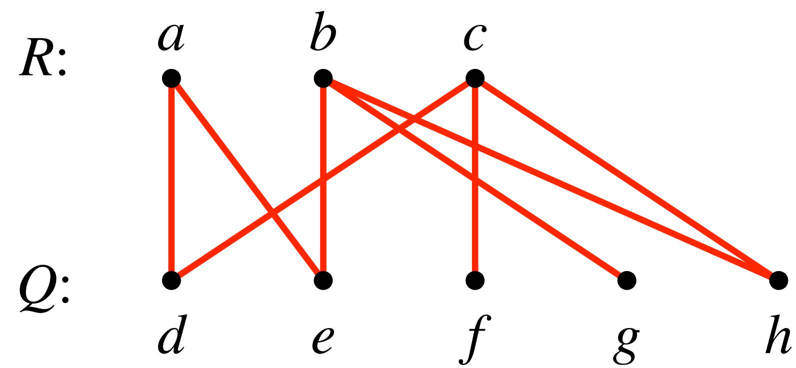

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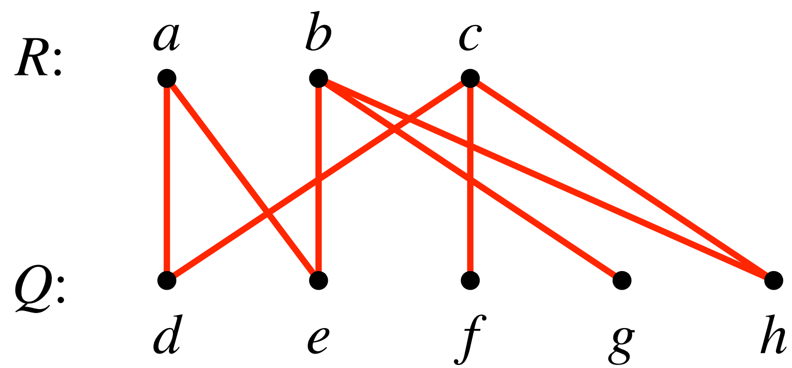

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$true \Rightarrow false$ is $false$.

Math Predicates

Even

Math Predicates

Even

even. x returns *true* iff x is an even number.

Math Predicates

<u>Even</u>

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Examples: even.12 returns $true$. even.3 returns $false$.

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12 is even because 6 exists. 10 is even because 5 exists.

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Odd

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Odd

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Formal definition of odd. x : $(\exists k : \mathbb{Z} \,|: 2 \cdot k + 1 = x)$

17 is odd because 8 exists. 15 is odd because 7 exists.

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Multiple of

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$\text{mult}(n, x)$ returns *true* iff n is a multiple of x.

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$\text{mult}(n,x)$ returns $true$ iff n is a multiple of x.

Examples:

$\text{mult}(27,9)$ returns $true$.

Math Predicates

Multiple of

mult(n,x) returns $true$ iff n is a multiple of x.

Examples:

mult$(27,9)$ returns $true$.

mult$(9,27)$ returns $false$.

Math Predicates

Multiple of

$\text{mult}(n,x)$ returns $true$ iff n is a multiple of x.

Examples:

$\text{mult}(27,9)$ returns $true$.

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Math Predicates

Multiple of

$\text{mult}(n,x)$ returns $true$ iff n is a multiple of x.

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$\text{mult}(27,9)$ returns $true$.

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Formal definition of $\text{mult}(n,x)$: $\quad (\exists k : \mathbb{Z} \mid : n = k \cdot x)$

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$\text{mult}(15,-3)$ returns $true$.

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27 is a multiple of 9 because 3 exists.

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Divides

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"x divides y" is written with the binary infix operator as $x \mid y$.

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$3 \mid 16 \equiv false$

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Alternate definition of $x \mid y$: $(\exists k : \mathbb{N} \mid : x \cdot k = y)$

Math Predicates

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"x divides y" is written with the binary infix operator as $x \mid y$.

$x \mid y$ returns $true$ iff x goes into y with no remainder and a non-negative quotient.

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$3 \mid 15 \equiv true$

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Formal definition of $x \mid y$: $\quad (\exists k : \mathbb{Z} \mid 0 \leq k : x \cdot k = y)$

Alternate definition of $x \mid y$: $\quad (\exists k : \mathbb{N} \mid : x \cdot k = y)$

-3 divides -15 because 5 exists and is non-negative.

Words for Universal Quantification

for all,   all,   every,   for each,   any,   a

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for all,   all,   every,   for each,   any,   a

Example

Words for Universal Quantification

for all, all, every, for each, any, a

Example

All even integers are multiples of two.

Words for Universal Quantification

for all, all, every, for each, any, a

Example

All even integers are multiples of two.

Every even integer is a multiple of two.

Words for Universal Quantification

for all, all, every, for each, any, a

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All even integers are multiples of two.
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<u>Example</u>

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Example

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Every even integer is a multiple of two.

Each even integer is a multiple of two.

Any even integer is a multiple of two.

An even integer is a multiple of two.

$(\forall x : \mathbb{Z} \mid \text{even}.\, x : \text{mult}(x, 2))$

Words for Universal Quantification

for all,   all,   every,   for each,   any,   a

Example

Words for Universal Quantification

for all, all, every, for each, any, a

Example

An integer greater than 200 is also greater than 100.

Words for Universal Quantification

for all, all, every, for each, any, a

Example

An integer greater than 200 is also greater than 100.

$(\forall x : \mathbb{Z} \mid x > 200 : x > 100)$

Words for Universal Quantification

for all, all, every, for each, any, a

Example

An integer greater than 200 is also greater than 100.

$(\forall x : \mathbb{Z} \mid x > 200 : x > 100)$

Example

Words for Universal Quantification

for all, all, every, for each, any, a

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An integer greater than 200 is also greater than 100.

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The square of an integer is non-negative.

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The square of an integer is non-negative.
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Words for Existential Quantification

exists, some, there are, there is, at least one

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<u>Example</u>

There exists an even integer that is a multiple of three.

Words for Existential Quantification

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Example

There exists an even integer that is a multiple of three.
Some even integer is a multiple of three.

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There are even integers that are multiples of three.
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At least one even integer is a multiple of three.
$(\exists x : \mathbb{Z} \mid \text{even}.\, x : \text{mult}(x, 3))$

Examples of Mathematical Quantification

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Exercise 9.29 (a)

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The natural number 1 is the only natural number that is smaller than positive integer p and divides p.

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